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Solvable Model with Interaction of Pair Dipoles in One Dimension

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We discuss the parity-conserved $\delta''(x)$ interaction between pair dipoles. It is shown that the case of zero coupling constant $c = 0$ is markedly different from that of $c \to 0$. This is an un-perturbation effect.

Two- and many-body problems with $\delta'(x)$ (first derivation of the Dirac δ-function) interaction are of great interest (Gesztesy and Holden, 1987; Seba, 1988). Pang *et al.* (1990) encountered this problem when studying the quantized Davey–Stewardson I system. Recently Zhao (1992) gave an interesting solution to the Schrödinger equation with two-body $\delta'(x)$ interaction. However, in one dimension, parity cannot be conserved in this system. So such a Schrödinger equation cannot properly describe the behavior of bosons or fermions in one dimension. We therefore consider the parity-conserved $\delta''(x)$ potential.

As is well known, the $\delta'(x)$ interaction is an idealization of forces within a point dipole,

$$\delta'(x) = \lim_{a \to 0} \frac{1}{a} \left[\delta\left(x + \frac{a}{2}\right) - \delta\left(x - \frac{a}{2}\right) \right] \tag{1}$$

whereas $\delta''(x)$ is similar to that between dipoles,

$$\delta''(x) = \lim_{a \to 0} \frac{1}{a^2} [\delta(x + a) - 2\delta(x) + \delta(x - a)] \tag{2}$$

It is possible that this is the force between electron pairs (or vacancy pairs) in extreme conditions such as low temperature. This model is therefore worthy of attention.

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First, we investigate the two-body Schrödinger equation,

$$-\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)\psi(x_1, x_2) + 4c\delta''(x_1 - x_2)\psi(x_1, x_2) = E\psi(x_1, x_2) \quad (3)$$

which can be reduced to a one-body problem in mass-centered coordinates (MCC),

$$-\frac{d^2}{dx^2}\psi(x) + 4c\delta''(x)\psi(x) = E\psi(x) \quad (4)$$

As usual, we require $\psi(x)$ to be continuous at $x = 0$. Integrating equation (4) from 0^- to 0^+, we obtain $\psi(0^+) - \psi(0^-) = 4c\psi''(0)$, where $\psi''(0)$ is defined as $[\psi''(0^+) + \psi''(0^-)]/2$. So equation (4) can be replaced by

$$\begin{aligned} -\frac{d^2}{dx^2}\psi(x) &= E\psi(x) \qquad \text{when} \quad x \neq 0 \\ \psi(0^+) &= \psi(0^-) = \psi(0) \\ \psi'(0^+) - \psi'(0^-) &= 4c\psi''(0) \end{aligned} \quad (5)$$

For bosons we have the symmetric solution

$$\psi(x) = \theta(x)(A_1 e^{ikx} + A_2 e^{-ikx}) + \theta(-x)(A_2 e^{ikx} + A_1 e^{-ikx})$$

where $k^2 = E > 0$, and $k = (k_2 - k_1)/2$ is the effective momentum in MCC. $\theta(x) = 0, 1/2, 1$ as $x <, =, > 0$. From (5) we have

$$2ik(A_1 - A_2) = -4ck^2(A_1 + A_2)$$

or

$$\frac{A_2}{A_1} = \frac{1 - 2ick}{1 + 2ick} = \frac{1 - ic(k_2 - k_1)}{1 + ic(k_2 - k_1)} = e^{i\Theta(k_2 - k_1)} \quad (6)$$

where $\Theta(x) = -2\tan^{-1}(cx)$.

It is not difficult to obtain a bound-state solution when $c < 0$,

$$\begin{aligned} \psi(x) &= e^{|x|/2c} \\ E &= -\frac{1}{4c^2} < 0 \end{aligned} \quad (7)$$

Second, we use the Bethe Ansatz (Shastry *et al.*, 1985) to analyze the many-body problem. The N-particle wave function can be written as

$$\psi(x_1, \ldots, x_N) = \sum_P \psi_{p_1 \ldots p_N}(x_1, \ldots, x_N)\theta(x_{p_1} < \cdots < x_{p_N}) \quad (8)$$

where $P = \{p_1, \ldots, p_N\}$ is a perturbation of $\{1, \ldots, N\}$. For bosons we have

$$\psi_{1,\ldots,N}(x_2, \ldots, x_N, x_1) = \psi_{2,\ldots,N,1}(x_1, \ldots, x_N), \quad \text{etc.} \tag{9}$$

Define

$$\psi_{1,\ldots,N}(x_1, \ldots, x_N) = \sum_P A_p \exp(ik_{p_1}x_1 + \cdots + ik_{p_N}x_N) \tag{10}$$

We can obtain other terms in (8). Furthermore, if the permutation P corresponds to momenta $\{k_{p_1}, \ldots, k_{p_N}\} = \{\ldots, k, k', \ldots\}$ and P' corresponds to $\{k_{p'_1}, \ldots, k_{p'_N}\} = \{\ldots, k', k, \ldots\}$, applying the boundary condition, we have the same result as in the two-particle case,

$$\frac{A_{P'}}{A_P} = e^{i\Theta(k'-k)} \tag{11}$$

Under the periodicity assumption and condition (9), using (11) $N-1$ times, we obtain

$$\exp(ik_i l) = \exp\left[i \sum_{j=1}^{N} \Theta(k_i - k_j)\right], \qquad i = 1, \ldots, N \tag{12a}$$

or

$$k_i l = 2\pi I_i + \sum_{j=1}^{N} \Theta(k_i - k_j), \qquad i = 1, \ldots, N \tag{12}$$

where I_i is an arbitrary integer. Thus we can solve the problem under any set of integers $\{I_i\}$. For every $\{k_i\}$ that satisfy equation (12), we have $E = \sum k_i^2$.

The results obtained above are similar to those for the $\delta(x)$ potential in some respects. However, there are significant and interesting differences.

1. Solutions of equation (4) under condition $c \to 0$ are different from those under $c = 0$. In particular, when $c \to 0^-$ and $E < 0$, there is a deeply bounded state. For positive E and $c \to 0$, the wave function can only be $\psi(x) = \cos(x)$, different from free particles. Because of all this, even when c is quite small, solutions to the Schrödinger equation can neither be neglected nor be obtained using the perturbation method.

2. Comparing equations (6) and (7) with that of the $\delta(x)$ interaction (Shastry *et al.*, 1985), we find $1/c$ is equivalent to the coupling constant of a $\delta(x)$ potential. Also, in the case of $\delta(x)$, the sign before $e^{i\Theta(k'-k)}$ is negative instead of positive in equation (6). So $\{I_i\}$ are all integers in equation (12), while we have half-integers for even numbers of N when considering the $\delta(x)$ interaction.

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